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Published in:

Journal of Physics A: Mathematical and Theoretical

DOI:

[10.1088/1751-8121/aa6243](https://doi.org/10.1088/1751-8121/aa6243)

Publication date:

2017

Citation for published version (APA):

Shackerley-Bennett, U., Pitchford, A., Genoni, M. G., Serafini, A., & Burgarth, D. (2017). The reachable set of single-mode quadratic Hamiltonians. *Journal of Physics A: Mathematical and Theoretical*, 50, [155203].
<https://doi.org/10.1088/1751-8121/aa6243>

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The reachable set of single-mode quadratic Hamiltonians

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(Dated: February 19, 2017)

Open-loop controllability in quantum mechanics refers to finding conditions on time-varying Hamiltonians such that a full group of unitary transformations can be enacted with them. For compact groups controllability is well understood and is dealt with using the Lie algebra rank criterion. Gaussian systems, however, evolve under Hamiltonians generating the non-compact symplectic group, rendering the rank criterion necessary but no longer sufficient. In this setting it is possible to satisfy the rank criterion without the ability to enact all symplectic transformations. We refer to such systems as ‘unstable’ and explore the set of symplectic transformations that remain reachable. We provide a partial analytical characterisation for the reachable set of a single-mode unstable system. From this it is proven that no orthogonal-symplectic operations (‘energy-preserving’ or ‘passive’ in the literature) may be reached with such controls. We then apply numerical optimal control algorithms to demonstrate a complete characterisation of the set in specific cases. These results suggest approaches to the long-standing open problem of controllability in n modes.

PACS numbers: 42.50.Dv, 03.65.-w, 02.30.Yy

I. INTRODUCTION

Achieving a high level of control over quantum mechanical systems is a central goal in modern physics. Many of these systems can be described using the Gaussian formalism – encompassing light fields [1], the motional degrees of freedom of trapped ions [2], opto- and nano-mechanical oscillators [3] and superconducting Josephson junctions [4]. Such systems allow for a variety of tasks such as entanglement generation, squeezing, cooling and quantum communication protocols [5, 6]. In these continuous variable regimes observables are functions of canonical mode operators, meaning that evolution is not determined by a finite-dimensional unitary group. Although control theory in regimes with general unbounded operators is being developed [7], here we will restrict to the subclass of Gaussian systems. This regime requires the Hamiltonians to be quadratic in the mode operators, causing the set of all possible unitary operations to form a finite-dimensional group, corresponding to the real symplectic group [8, 9].

Control theory calls us to imagine a time-varying Hamiltonian generating a trajectory on the group manifold via the exponential map. Controllable systems are those in which every element of the group is reachable along such a trajectory. This property is well understood for compact groups including, for example, the set of finite-dimensional unitary matrices of a given dimension. To state a condition on the Hamiltonian we consider the time-varying version as a set of constant Hamiltonians. A necessary and sufficient condition for this set to form a controllable system is that its elements generate the Lie algebra of the group by linear combination and repeated commutation; this is referred to as the Lie algebra rank criterion [10]. For the symplectic group this criterion is no longer sufficient [11] due to its non-compact nature, providing the trajectories with the possibility of not recurring, this being impossible on compact groups. This characteristic presents itself in the operation known as squeezing in quan-

tum optics, which can proceed indefinitely without recurrence to identity. Intuitively, squeezing can be seen as a symplectic transformation which reduces the uncertainty on one canonical observable while increasing it on the other, maintaining the Heisenberg uncertainty relation. There is no limit to how far this may proceed as long as the uncertainty relation is satisfied.

In a seminal paper, Jurdjevic and Sussmann [12] proved a sufficient condition for controllability on non-compact Lie groups by considering the existence of control Hamiltonians that recur. This aspect has also been noted in the context of quantum optics where recurrence is associated with a positive control Hamiltonian [13]. The recurring nature of a control Hamiltonian is elsewhere referred to as the property of ‘neutrality’ [14].

It can be shown that the neutrality condition is both necessary and sufficient for the controllability of single-mode systems [15]. For systems without such a Hamiltonian one may engineer situations where the Lie algebra rank criterion holds and yet the reachable set is not the whole symplectic group. In quantum optical language these systems whose control Hamiltonians have a squeezing component that is strong enough to prevent recurrence. Since all such Hamiltonians are unbounded from below, we shall refer to such systems as ‘unstable’.

In this paper we apply analytical and numerical techniques to investigate unstable single-mode systems. The springboard for the analysis will be the results of Wu, Li, Zhang and Tarn [15] concerning the uncontrollability of such systems. Note that in that paper they consider physical systems evolving according to the group $SU(1, 1)$, rather than our symplectic $Sp_{2, \mathbb{R}}$, defined in Sec. II. It can be shown that these groups are isomorphic and so the results apply in both cases. $SU(1, 1)$ is central in the dynamics of such systems as Bose-Einstein condensates [16] and spin wave transitions in solid-state physics [17].

We find that in single-mode unstable systems unbounded

squeezing is the mechanism behind uncontrollability. To the authors' knowledge this is the first time that this physical mechanism has been formally related to the question of controllability of Gaussian systems. Furthermore, we demonstrate the application of optimal control techniques to the symplectic group, first explored in [18], and complete the characterisation of the reachable set through the resulting numerics. These results open up a physical understanding of control theory with the proof that non-trivial, energy-preserving symplectics, which correspond to phase-shifters in the lab, are unreachable for single-mode unstable systems. This suggests possible routes for the discovery of a necessary and sufficient condition in any number of modes.

II. PRELIMINARIES AND SETUP

To introduce the control theory of continuous variable systems we start with the Heisenberg equation,

$$\dot{\hat{\rho}} = i[\hat{H}, \hat{\rho}], \quad (1)$$

solved by the ansatz,

$$\hat{\rho}(t) = \hat{U}\hat{\rho}(0)\hat{U}^\dagger, \quad (2)$$

for unitary \hat{U} provided

$$\dot{\hat{U}} = i\hat{H}\hat{U}. \quad (3)$$

In control theory \hat{H} is a function of time, allowed to access a subset of all possible Hamiltonians for the system. The question then arises as to whether all elements of the corresponding unitary group are solutions to, or 'reachable under', Eq. (3). If the answer is positive then the system is referred to as operator-controllable [10], or simply controllable for the purposes of this paper.

To explore systems in which $\hat{\rho}$ is a Gaussian state we must first confine the properties of the Hamiltonian. Let $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^\top$ be a vector of canonical operators such that $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$. One refers to a Hamiltonian \hat{H} as quadratic if it can be written as $\hat{H} = \frac{1}{2}\hat{\mathbf{r}}^\top H \hat{\mathbf{r}}$, where H is a real, symmetric $2n \times 2n$ matrix. We may exponentiate these quadratic Hamiltonians to form a unitary representation of the metaplectic group [19]. This group is defined as the double cover of the symplectic group $\text{Sp}_{2n, \mathbb{R}}$, which is the set of matrices S such that

$$S\Omega S^\top = \Omega, \quad \text{where} \quad \Omega := \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4)$$

It turns out that the distinction between metaplectic and symplectic involves a sign which plays no part in the dynamics of continuous variable systems. As such, when exploring unitary control of these systems it suffices to consider control on the symplectic group. It is possible to show that the symplectic matrix transformation corresponding to a quadratic Hamiltonian enacted for time t is given by $S = e^{\Omega H t}$. This hints

correctly that the Lie algebra of the symplectic group $\mathfrak{sp}_{2n, \mathbb{R}}$ is given by the set of matrices of the form ΩH , where H has the properties stated above.

As a result we may translate Eq. (3) into the form

$$\dot{S} = X S, \quad (5)$$

where $X = \Omega H$ depends on time. A standard approach is to consider X as consisting of two parts: an always-on 'drift term' A and a constant control term B with a coefficient varying in \mathbb{R} [11]. Restricting to the single-mode case ($n = 1$) this transforms Eq. (5) to

$$\dot{S}(t) = (A + u(t)B) S(t), \quad S(0) = \mathbb{I}_2, \quad (6)$$

where the control function $u(t)$ is any locally bounded measurable function defined on the positive time domain $[0, \infty)$ with $A, B \in \mathfrak{sp}_{2, \mathbb{R}}$. Given A and B , the subset Ξ of $\mathfrak{sp}_{2, \mathbb{R}}$ with elements of the form $A + vB$, $v \in \mathbb{R}$, is called the set of accessible dynamical generators of the system.

It may be noted that we could consider more than one control term for complete generality. However, when we look at the behaviour of such cases it will become apparent that they are not interesting. We could have included a maximum of three linearly independent control terms but if that were the case then the drift term would be subsumed in the controls. Using two independent control generators, instead, would either subsume the drift term or imply that Ξ contains a recurring element that would imply controllability [15]. In all such cases the Lie algebra rank criterion would again become sufficient for controllability. Here, we would like to characterise systems for which this condition is satisfied but *not* sufficient and therefore we consider a single control term B .

In order to characterise which symplectic operations will be achievable for a certain Ξ we define the reachable set as follows (as customary in control theory):

Definition: *Reachable set.* The union of all sets of elements $S(t)$ that solve Eq. (6) for some choice of control function $u(t)$ is called the reachable set of Eq. (6) and is denoted \mathcal{R} .

When considering non-compact groups it becomes important to understand which types of generator could be contained in Ξ . The elements of M of $\mathfrak{sp}_{2, \mathbb{R}}$ are often categorised as

- Parabolic, if $\text{Tr}[M^2] = 0$,
- Hyperbolic, if $\text{Tr}[M^2] > 0$,
- Elliptic, if $\text{Tr}[M^2] < 0$.

In the physical picture, a trajectory set by a single-mode elliptic generator is a recurring, 'stable' one, corresponding to a strictly positive or strictly negative H . This fact can quickly be seen by noting that $\text{Tr}[M^2] \propto -\text{Det}[M] \propto -\text{Det}[H]$, using $M = \Omega H$. Explicitly $S = e^{M t}$ will get arbitrarily close to the identity, in any matrix topology, at some positive time t . This latter condition is precisely the condition of neutrality and so the two meanings coincide for single-mode systems. For multimode systems the definitions diverge and neutrality becomes the property of study.

The first aim is to characterise Ξ such that its elements satisfy the Lie algebra rank criterion, i.e. that its elements generate $\mathfrak{sp}_{2,\mathbb{R}}$, but do not allow \mathcal{R} to be equal to $\text{Sp}(2, \mathbb{R})$. As already stated, given the rank criterion, neutrality is sufficient for controllability and so this immediately removes elliptic elements from Ξ . It may therefore seem obvious that all such systems are uncontrollable since it is easy to show that the set e^X for all X non-elliptic is not $\text{Sp}(2, \mathbb{R})$. However, this would be to forget the fact that the time variation in Eq. (6) allows the symplectics to be reached by products of exponentials. Given the fact that hyperbolic generators can generate the whole algebra, it is less clear that this should be so.

On the other hand, both hyperbolic and parabolic generators do not possess the recurring properties of the elliptics. It is possible to show, however, that when Ξ contains parabolic elements the rank criterion is either not satisfied or the system is controllable [15]. Therefore we consider Ξ with only hyperbolic generators.

To complete the setup let us also specify a basis of $\mathfrak{sp}_{2,\mathbb{R}}$:

$$K_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_z = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (7)$$

which satisfies the algebra

$$[K_x, K_y] = -K_z, \quad [K_y, K_z] = K_x, \quad [K_z, K_x] = K_y. \quad (8)$$

The generator K_z is elliptic and the group elements $e^{K_z t}$ are the 2-dimensional rotations forming the Abelian subgroup $\text{SO}(2)$: the set of 2×2 , orthogonal matrices with unit determinant. This turns out to be the maximal compact subgroup of $\text{Sp}_{2,\mathbb{R}}$. In the lab, these correspond to phase-plates or phase-shifters that rotate the optical phase of a field. These are also known as ‘passive’ or ‘energy-preserving’, in that they preserve the number of field excitations. The hyperbolic generators K_x and K_y instead generate single-mode squeezing operations. If a linear combination of generators is considered, such as $aK_y + K_z$, with $a \in \mathbb{R}$, one can show that it is elliptic for $|a| < 1$, parabolic for $|a| = 1$, and hyperbolic for $|a| > 1$.

III. UNCONTROLLABILITY OF UNSTABLE SYSTEMS

In this paper we consider systems where the rank criterion is satisfied but where the entire group is not reachable. To study such systems it suffices to consider those for which the elements of Ξ are purely hyperbolic. Before moving into the characterisation of reachable sets it is necessary to review some results from existing controllability analysis. We follow the treatment of [15] faithfully and, in order to make the paper self-contained, reproduce the proofs of these statements in Appendix A.

It is easy to show, by example, that Ξ containing only hyperbolic elements is capable of generating $\mathfrak{sp}_{2,\mathbb{R}}$. Therefore fixing Ξ to be the set of all hyperbolic elements implies that the rank criterion will be satisfied.

Lemma 1. *If Ξ only contains hyperbolic elements then*

Eq. (6) is similar, via a symplectic transformation, to

$$\dot{S}(t) = (-K_x + bK_z + u(t)K_y)S(t), \quad S(0) = \mathbb{I}_2, \quad (9)$$

where b is some real constant with modulus strictly less than one.

The reachable set of such a system will be symplectically similar to the reachable set of any purely hyperbolic system and so this result allows us to study one and then draw conclusion about them all. We denote the set of accessible controls of Eq. (9) by $\tilde{\Xi}$, with elements of the form $-K_x + bK_z + vK_y$, $v \in \mathbb{R}$, and its reachable set by $\tilde{\mathcal{R}}$.

In order to state the following lemma we consider a general 2×2 real matrix written as

$$X = \begin{pmatrix} x_1 + x_3 & x_2 + x_4 \\ x_4 - x_2 & x_1 - x_3 \end{pmatrix}, \quad x_i \in \mathbb{R}. \quad (10)$$

Lemma 2. *If $X \in \tilde{\mathcal{R}}$ then the function*

$$f(X) := (x_1 - x_4)^2 - (x_2 - x_3)^2 \quad (11)$$

satisfies

$$\begin{aligned} f(X) &\geq 1, \\ \dot{f}(X) &\geq 0, \\ \dot{f}(X = \mathbb{I}) &\geq 1, \end{aligned} \quad (12)$$

for any choice of $u(t)$ in Eq. (9).

There exists $X \in \text{Sp}_{2,\mathbb{R}}$ such that $f(X) < 1$ and a symplectic transformation is unable to turn a subset $\text{Sp}_{2,\mathbb{R}}$ into the whole group. Putting these results together it is concluded that systems with Ξ purely hyperbolic systems are not controllable. From these results we may proceed to a quantum optical characterisation.

IV. SINGULAR VALUE DECOMPOSITION

In order to set our findings against the backdrop of quantum optics it is advantageous to introduce the singular value decomposition for symplectic transformations and to take some care in defining its elements uniquely. The singular value decomposition of symplectic matrices is often referred to as the Euler [19] or Bloch-Messiah [20] decomposition in the literature, and its form is easily related to physical implementations. Each n -mode symplectic matrix can be decomposed into the product of two passive operations, belonging to the intersection between $\text{Sp}_{2n,\mathbb{R}}$ and $\text{SO}(2n)$, and a direct sum of diagonal squeezing operations. In quantum optical implementations, passive operations correspond to beam splitters and phase-plates, which do not alter the energy of the free field. In a single mode this intersection simply gives $\text{SO}(2)$.

Here, we define the singular value decomposition indicating the necessary bounds for uniqueness in one mode. This uniqueness is vital in visualising the reachable set.

Definition: *Singular value decomposition.* Define

$$\text{SO}(2) := \left\{ \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \quad (13)$$

and

$$\mathcal{Z}(2, \mathbb{R}) := \{\text{diag}(1/z, z) \mid z \in \mathbb{R}, z \geq 1\}. \quad (14)$$

Any $X \in \text{Sp}_{2, \mathbb{R}}$ can be decomposed as either

$$X = R_\theta Z R_\phi \quad \text{or} \quad X = R_\theta, \quad (15)$$

where $R_\theta, R_\phi \in \text{SO}(2)$ and $Z \in \mathcal{Z}(2, \mathbb{R})$. For the singular value decomposition to be unique, the allowed angles must be bounded such that

$$-\pi + \theta_0 \leq \theta < \pi + \theta_0, \quad -\frac{\pi}{2} + \phi_0 \leq \phi < \frac{\pi}{2} + \phi_0, \quad (16)$$

where θ_0 and ϕ_0 are arbitrary but fixed. See Appendix B 1 for a justification of these bounds.

The singular value decomposition allows one to characterise symplectic matrices in terms of the new parameters θ , ϕ and z . Thus Lemma 2 will have a new form in these coordinates. The coordinate transformation is derived in Appendix B 2 and the restatement of Lemma 2 is given here:

Corollary 1. *If $X \in \tilde{\mathcal{R}}$ then the function*

$$g(X) := \cos[2\theta] \cos[2\phi] - \lambda(z, \phi) \sin[2\theta], \quad (17)$$

where

$$\lambda(z, \phi) := \frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) \sin[2\phi] - \frac{1}{2} \left(z^2 - \frac{1}{z^2} \right), \quad (18)$$

satisfies

$$g(X) \geq 1, \quad (19)$$

$$\dot{g}(X) \geq 0, \quad (20)$$

$$\dot{g}(X = \mathbb{I}) \geq 1. \quad (21)$$

for any choice of $u(t)$ in Eq. (9).

With this we are now in a position to develop a physical realisation of the form of $\tilde{\mathcal{R}}$.

V. REACHABLE SETS OF UNSTABLE SYSTEMS

To gain some analytical insight into unstable systems we use Corollary 1 to provide some bounds on $\tilde{\mathcal{R}}$.

Lemma 3. *The existence of solutions for $g(X) > d$, where $d \geq 1$, implies that*

$$z > \sqrt{\frac{d+1}{2}}. \quad (22)$$

Proof First we prove that $g(X) > 1$ implies that $\sin[2\theta] \geq 0$. Define

$$\delta := \lambda(z, \phi) - \sin[2\phi], \quad (23)$$

which allows one to rewrite $g(X)$ as

$$g(X) \equiv \cos[2(\theta - \phi)] - \delta \sin[2\theta]. \quad (24)$$

If $\sin[2\theta] < 0$ then $g(X) > 1$ only has solutions if $\delta > 0$. This is only true if $\lambda(z, \phi) > \sin[2\phi]$. Hence

$$\frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) \sin[2\phi] - \frac{1}{2} \left(z^2 - \frac{1}{z^2} \right) > \sin[2\phi], \quad (25)$$

equally

$$\left(z^2 + \frac{1}{z^2} - 2 \right) \sin[2\phi] > z^2 - \frac{1}{z^2}. \quad (26)$$

$z^2 + 1/z^2 - 2$ is positive for all values of z and $\sin[2\phi] \leq 1$. Therefore this has only solutions if

$$z^2 + \frac{1}{z^2} - 2 > z^2 - \frac{1}{z^2}, \quad (27)$$

implying $z < 1$ which we have ruled out by convention. Proving that $\sin[2\theta] \geq 0$.

Now we look for existence of solutions to the inequality $g(X) > d$. Because $\sin[2\theta] \geq 0$ these exist if and only if there exist solutions to

$$\delta < -(d-1). \quad (28)$$

This translates to

$$\frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) \sin[2\phi] - \frac{1}{2} \left(z^2 - \frac{1}{z^2} \right) < \sin[2\phi] - (d-1), \quad (29)$$

equally

$$(z^2 + \frac{1}{z^2} - 2) \sin[2\phi] < z^2 - \frac{1}{z^2} - (d-1). \quad (30)$$

$z^2 + 1/z^2 - 2$ is positive for all values of z and $\sin[2\phi] \leq 1$. Therefore this has solutions if and only if

$$z^2 + \frac{1}{z^2} - 2 < z^2 - \frac{1}{z^2} - (d-1), \quad (31)$$

which only has solutions for

$$z^2 > \frac{d+1}{2}, \quad (32)$$

proving the statement. \square

Lemma 3 links lower bounds on $g(X)$ to lower bounds on z which will be used to further understand the boundary of the reachable set when considered in (z, θ, ϕ) space.

Lemma 4. *There does not exist $X \in \tilde{\mathcal{R}}$ such that*

$$X = S R_\theta S^{-1}, \quad (33)$$

where $S \in \text{Sp}_{2, \mathbb{R}}$, $R_\theta \in \text{SO}(2)$.

Proof Assume there exists $X \in \tilde{\mathcal{R}}$ that satisfies the above condition. We know that

$$X^m \in \tilde{\mathcal{R}} \quad \forall m \in \mathbb{N} \quad (34)$$

because the reachable set of Eq. (9) has a semigroup structure. Note that

$$\begin{aligned} \|X^m - \mathbb{I}\| &= \|S(R_\theta^m - \mathbb{I})S^{-1}\| \\ &\leq \|S\| \|S^{-1}\| \|R_\theta^m - \mathbb{I}\|, \end{aligned} \quad (35)$$

where we use the Euclidean norm

$$\|X\| := \sqrt{\text{Tr}[X^\top X]}. \quad (36)$$

S is time-independent and so $\|S\| \|S^{-1}\|$ is constant. R_θ is quasi-periodic and so there must exist some m such that

$$\|R_\theta^m - \mathbb{I}\| < \varepsilon, \quad \forall \varepsilon > 0 \quad (37)$$

and so there exists m such that

$$\|X^m - \mathbb{I}\| < \varepsilon, \quad \forall \varepsilon > 0. \quad (38)$$

From Eq. (20) we know that the value of $g(X)$ must be non-decreasing along any trajectory of the system and from Eq. (21) we know that its rate of change at identity is 1. As a result, for some finite evolution time of Eq. (9) all subsequent trajectories must contain elements that have a lower bound on their value of $g(X)$ that is greater than 1. By Lemma 3 this implies a lower bound on the value z along a given trajectory of the control system given some minimal evolution time. X^m is a possible trajectory of the system for all m and we can find m such that the z value of X^m is arbitrarily close to 1 violating the lower bound. Therefore such an X cannot be an element of $\tilde{\mathcal{R}}$. \square

Theorem 1. *If Eq. (6) is restricted to hyperbolic dynamical generators then its reachable set does not contain any elements of $\text{SO}(2)$ except for \mathbb{I} .*

Proof The reachable set \mathcal{R} of Eq. (6) is symplectically similar to $\tilde{\mathcal{R}}$. Lemma 4 states that $\tilde{\mathcal{R}}$ does not contain any element that is symplectically similar to an element of $\text{SO}(2) \setminus \mathbb{I}$. Thus \mathcal{R} does not contain any element of $\text{SO}(2) \setminus \mathbb{I}$. \square

In practice this result implies that no manipulation in time of the control functions ever allows one to achieve an optical phase-shift operation for unstable systems. This holds even if the control Hamiltonians are able to generate the whole symplectic algebra. The behaviour of single-mode hyperbolic systems seems to imply an ever increasing squeezing value that invokes intuition for potentially similar behaviour in the multimode case.

Note again that the exclusion of elliptic elements from Ξ does not immediately yield this result as it is conceivable that one could squeeze and then unsqueeze to a passive operation without ever needing an elliptic dynamical generator.

VI. EXAMPLE SYSTEM: CONTROLLED SQUEEZING HAMILTONIANS

In the following we consider the reachable set of a specific unstable system. In the single-mode scenario, such a system

can be obtained by taking drift and control Hamiltonians as squeezing operations along different directions. In the Hilbert space picture we consider

$$\hat{H} = \hat{H}_A + u(t) \hat{H}_B, \quad (39)$$

where the drift Hamiltonian \hat{H}_A and the control Hamiltonian \hat{H}_B are defined as

$$\begin{aligned} \hat{H}_A &= \frac{(1-c)\hat{x}^2 - (1+c)\hat{p}^2}{2}, \\ \hat{H}_B &= -\frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}. \end{aligned} \quad (40)$$

The theory of Sec. II allows us to translate these to

$$H_A = \begin{pmatrix} 1-c & 0 \\ 0 & -c-1 \end{pmatrix}, \quad H_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (41)$$

This in turn sets up the the open-loop control problem

$$\dot{S}(t) = (A + u(t)B)S(t), \quad (42)$$

where

$$A = \begin{pmatrix} 0 & -(1+c) \\ -(1-c) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (43)$$

The reason for introducing this example system is to visualise its reachable set in order to give intuition for its structure.

Note that A is parabolic, hyperbolic or elliptic if, respectively, $|c|$ is equal to, less than, or greater than 1. We focus our analysis on $c = 0$ to explore a hyperbolic case but later remark on changes of behaviour as we vary c . Before proceeding to the numerics we require a little more theoretical preparation.

A. Visualisation setup

The visualisation will use the singular value decomposition defined in Sec. IV and use (z, θ, ϕ) as graph axes. In order to plot the reachable set uniquely it is necessary to fix θ_0 and ϕ_0 . We choose

$$(\theta_0, \phi_0) = (0, \frac{\pi}{2}), \quad (44)$$

which implies the bounds

$$-\pi \leq \theta < \pi, \quad 0 \leq \phi < \pi. \quad (45)$$

We will represent the reachable set as points in a cubic space with the θ and ϕ ranges as given. One may object that, although for $z > 1$ this provides a unique mapping to the graph, for $z = 1$ this will not be the case because these matrices should be indicated by *one* and not *two* parameters, as per Eq. (15). By Theorem 1 we know that none of the elements of this plane will be reachable except for identity and so, after finding a point for identity, we may maintain the cubic plot for illustrative clarity.

Analytically, we may find the ‘singular value decomposition of identity’ by considering the limit $t \rightarrow 0$ for some reachable element of our example system. Take, for example,

$$\exp \left[\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} t \right] \quad (46)$$

as $t \rightarrow 0$. Consider $t = \frac{1}{n}$, where $n \in \mathbb{N}$.

$$\begin{aligned} \exp \left[\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{1}{n} \right] &= \left(R_{-\frac{3\pi}{4}} \begin{pmatrix} \frac{1}{e} & 0 \\ 0 & e \end{pmatrix} R_{\frac{3\pi}{4}} \right)^{\frac{1}{n}} \\ &= R_{-\frac{3\pi}{4}} \begin{pmatrix} \frac{1}{e} & 0 \\ 0 & e \end{pmatrix}^{\frac{1}{n}} R_{\frac{3\pi}{4}}. \end{aligned} \quad (47)$$

In the limit as $n \rightarrow \infty$ we find that the singular value decomposition of the identity is singled out as

$$\mathbb{I} = R_{-\frac{3\pi}{4}} R_{\frac{3\pi}{4}}. \quad (48)$$

Using this result, we may also derive a bound on the angle θ that will appear in the numerics. Analysis given in the proof of Lemma 3 states that for unstable systems $\sin[2\theta] > 0$ for $S \neq \mathbb{I}$ and so, given the range as set by Eq. (45), this implies

$$-\pi < \theta < -\frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}. \quad (49)$$

Eq. (48) indicates that at $t = 0$, $\theta = -3\pi/4$. The singular value decomposition of elements must vary continuously and therefore

$$-\pi < \theta < -\frac{\pi}{2}. \quad (50)$$

The numerics reported in the following subsection confirm and extend this analytical characterisation.

B. Numerical study through optimal control

Here we complement our analytics by applying optimal control algorithms adapted to the symplectic case in order to explore the reachable set of Eq. (42). We look to determine whether specific symplectic transformations S_{target} can be performed on our system given a fixed evolution time T . To test for controllability we implemented specific modules for simulating control in symplectic systems into QuTiP, which is an open source python library for simulating quantum dynamics [21, 22]. The GRAPE algorithm [23] is used to find a control function $u(t)$ that will drive the system to perform the transformation S_{target} . The evolution time T is split into Q equal time slices of length Δt with the time at the beginning of each slice t_k . $u(t_k)$ is constant throughout the time slice, hence the piecewise constant control function $u(t)$ corresponds to a set of Q real values. In this case $Q = 10$.

The dynamical generators used in QuTiP are of the form

$$H_k = H_A + u(t_k)H_B, \quad u(t_k) \in \mathbb{R}. \quad (51)$$

where H_A and H_B are as given in Eq. (41).

The evolution in each time slice is given by

$$S_k = e^{\Omega H_k \Delta t}. \quad (52)$$

The full evolution is given by

$$S(T) = S_Q S_{Q-1} \cdots S_k \cdots S_2 S_1. \quad (53)$$

The difference between the evolved transformation and the target is quantified by the fidelity error (or infidelity) as measured by the Frobenius norm

$$\varepsilon := \lambda \text{Tr}[(S(T) - S_{target})^\top (S(T) - S_{target})], \quad (54)$$

with $\lambda = 1/8$ for a 2×2 matrix.

The control function is optimised to minimise ε using the L-BFGS-B method in the *scipy* optimization function, which is a wrapper to the implementation by Byrd *et al.* [24]. The exact gradient with respect to $u(t_k)$ is calculated using the Frechet derivative (or augmented matrix method) as described in Eq. (12) of [25]. The target is considered achieved in this case if $\varepsilon < 10^{-3}$. The control function optimisation terminates unsuccessfully if either a local minima is found or a processing time limit is exceeded.

The set of possible target symplectics is discretised in the (z, θ, ϕ) space by only considering points at $\pi/12$ intervals in the angular directions and 10 logarithmically equal intervals between $z = 1$ and the arbitrary upper bound of $z = 100$. A bisection method was used to determine the boundary between reachable and unreachable targets. The boundary points are depicted as the darker blue points in Fig. 1. These simulations were repeated for combinations of $c = \{0.0, \pm 0.5, \pm 0.9, \pm 0.99, \pm 1.01, \pm 1.1, \pm 1.5\}$ and $T = \{0.1, 0.5, 1, 2, 3, 4, 5, 7, 10, 20, 50, 100\}$. When successful, the S_{target} test shows that there is at least one set of controls that can achieve the target transformation.

The results of some of the tests for unstable systems are shown in Fig. (1). Note that the points shown are those reachable specifically at evolution time T rather than up to time T . For unstable systems the reachable points are restricted to a set centred around $(\theta, \phi) = (-3\pi/4, 3\pi/4)$ and bounded by $-\pi < \theta < -\pi/2$, confirming the analytics, and $\pi/2 < \phi < \pi$, which was not proved analytically. This indicates that the numerics supply a tighter bound.

The example system is demonstrated to be unstable for $-1 < c < 1$. For $|c| \geq 1.1$ all points were found to be reachable. For $|c| = 1.01$ the optimiser was unable to find a suitable control function for some S_{target} . These unreachable points were predominantly in the region found reachable for $-1 < c < 1$. However, it is most likely that this is due to the constraints placed on the pulse optimisation, and demonstrates the difficulty of finding a solution near the edge of stability. Fig. (1) shows the case for $c = -0.99$ where we see that the reachable set is broader. There is then a discontinuity as we pass $|c| = 1$ when the reachable set then becomes the whole space. The broadening of the reachable set as c goes near the boundary indicates that the control system has become in a sense more stable.

The numerics show that, in one mode, when an elliptic drift field cannot be constructed, the system will restrict itself to

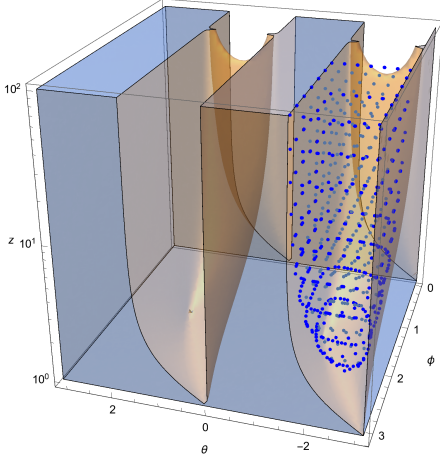
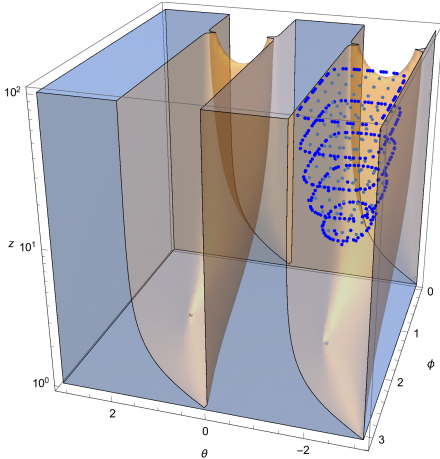
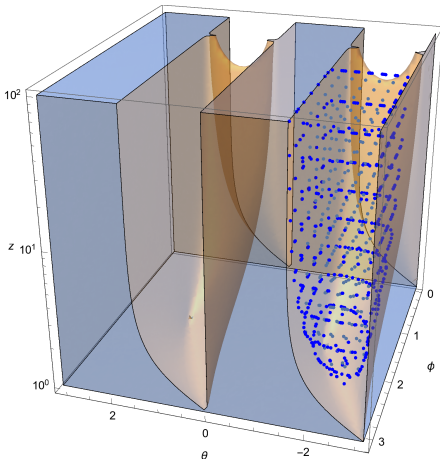
(a) $c = 0, T = 1$ (b) $c = 0, T = 5$ (c) $c = -0.99, T = 5$ 

FIG. 1. The blue points are reachable operations in the (z, θ, ϕ) basis after time T . The enclosed blue region is unreachable by Eq. (19). Specific parameters: (a) $c = 0, T = 1$; (b) $c = 0, T = 5$; (c) $c = -0.99, T = 5$.

unbounded squeezing within a small angular region. The ability to visualise this behaviour is by virtue of working in a single mode and a generalisation of this would require a more sophisticated treatment. Nevertheless, working on the numerics for this case provides some much needed intuition for a higher mode exploration.

VII. CONCLUSIONS AND OUTLOOK

The Lie algebra rank criterion, although necessary and sufficient for controllability on compact Lie groups, loses its sufficiency when the group is non-compact. Such a group occurs when considering the control of Gaussian quantum systems.

In this paper we sought to characterise and visualise single-mode systems that obey the rank criterion but are not controllable. The main aim was to describe the physical characteristics that prevent the criterion extending its sufficiency to non-compact Lie groups. In this process we found that strong squeezing in a particular direction on the group manifold provided the mechanism for uncontrollability. It was not obvious that no non-trivial passive, or energy-preserving, operation would be unreachable and this was proven. If this extended to unstable system in any number of modes then a long-standing open problem in mathematical control theory would be solved [12, 26].

Note that controlled operations generated by hyperbolic generators are accessible in several experimental set-ups, both optical and mechanical. Given the exponential speed-up they grant they are instrumental in beating decoherence times – for example [27], where such operations are proposed to achieve this aim in the context of superconducting quantum magnetomechanics, and the Hamiltonians generating them are referred to as ‘repulsive potentials’.

In multimode systems we expect to see similar behaviour. The crucial recurring elements are referred to as neutral and it is known that these are sufficient, with the rank criterion, for controllability on n modes [14]. To prove their necessity it would suffice to show that without them there is no way of accessing the maximal compact subgroup and that they induce a permanent state of squeezing. This is a new line of enquiry that will be developed from this physical characterisation of the single-mode case. It is hoped that this will provide further insight into the more general mathematical problem of controllability of closed, continuous variable, quantum systems.

VIII. ACKNOWLEDGEMENTS

We thank Ugo Boscain, Mario Sigalotti and Pierre Rouchon for discussions (during which Ugo pointed us to reference [15]). MGG and AS acknowledge financial support financial support from EPSRC through grant EP/K026267/1. MGG acknowledges support from the Marie Skłodowska-Curie Action H2020-MSCA-IF-2015. DB acknowledges support from EPSRC grant EP/M01634X/1. We are grateful to HPC Wales for giving access to the cluster that was used to perform the numerical simulations.

Appendix A: Complete proof of the uncontrollability condition

In order to prove Lemmata 1 and 2, we need a few preliminary statements, which are also taken directly from [15].

Lemma 5. *The equation*

$$\text{Tr}[[M, N]^2] = \text{Tr}[MN]^2 - 2 \text{Tr}[N^2] \text{Tr}[M^2] \quad (\text{A1})$$

holds for $M, N \in \mathfrak{sp}_{2, \mathbb{R}}$.

Proof First we expand the elements in the basis defined in Eq. (7):

$$M = m_1 K_x + m_2 K_y + m_3 K_z, \quad (\text{A2})$$

$$N = n_1 K_x + n_2 K_y + n_3 K_z. \quad (\text{A3})$$

We use this expansion to express the value of the following terms:

$$\text{Tr}[M^2] = \frac{1}{2}(m_1^2 + m_2^2 - m_3^2), \quad (\text{A4})$$

$$\text{Tr}[N^2] = \frac{1}{2}(n_1^2 + n_2^2 - n_3^2), \quad (\text{A5})$$

$$\text{Tr}[MN] = \frac{1}{2}(m_1 n_1 + m_2 n_2 - m_3 n_3), \quad (\text{A6})$$

$$\begin{aligned} \text{Tr}[[M, N]^2] &= \frac{1}{2}((m_2 n_3 - m_3 n_2)^2 \\ &\quad + (m_3 n_1 - m_1 n_3)^2 \\ &\quad - (m_1 n_2 - m_2 n_1)^2). \end{aligned} \quad (\text{A7})$$

Then we combine them to prove the statement:

$$\text{Tr}[[M, N]^2] = \text{Tr}[MN]^2 - 2 \text{Tr}[N^2] \text{Tr}[M^2]. \quad (\text{A8})$$

□

Lemma 6. *If $\text{Tr}[[A, B]^2] = 0$ in Eq. (6) then the system does not obey the Lie algebra rank criterion.*

Proof From Eqs. (A2) and (A3) it can be concluded that M , N and $[M, N]$ are linearly dependent if and only if

$$\text{Det} \begin{pmatrix} m_1 & n_1 & m_2 n_3 - m_3 n_2 \\ m_2 & n_2 & m_3 n_1 - m_1 n_3 \\ m_3 & n_3 & -(m_1 n_2 - m_2 n_1) \end{pmatrix} = 0, \quad (\text{A9})$$

or equivalently

$$(m_2 n_3 - m_3 n_2)^2 + (m_3 n_1 - m_1 n_3)^2 - (m_1 n_2 - m_2 n_1)^2 = 0. \quad (\text{A10})$$

From Eq. (A7) we see that this is equivalent to $\text{Tr}[[M, N]^2] = 0$. If A , B and $[A, B]$ are linearly dependent then the span of A and B does not generate $\mathfrak{sp}_{2, \mathbb{R}}$. □

Lemma 7. *Consider hyperbolic $M \in \mathfrak{sp}_{2, \mathbb{R}}$. There exists $P \in \text{Sp}_{2, \mathbb{R}}$ such that $PMP^{-1} = \sqrt{2 \text{Tr}[M^2]} K_y$.*

Proof M is hyperbolic and so $\text{Tr}[M^2] > 0$. We seek a matrix $P_1 = e^{\alpha K_z} \in \text{Sp}_{2, \mathbb{R}}$ which satisfies

$$P_1 M P_1^{-1} = \sqrt{m_1^2 + m_2^2} K_y + m_3 K_z. \quad (\text{A11})$$

using the decomposition of Eq. (A2). Let α be the angle satisfying

$$\sin[\alpha] = \frac{m_1}{\sqrt{m_1^2 + m_2^2}}, \quad \cos[\alpha] = \frac{m_2}{\sqrt{m_1^2 + m_2^2}}. \quad (\text{A12})$$

According to the formula

$$e^M N e^{-M} = N + [M, N] + \frac{1}{2!}[M, [M, N]] + \dots, \quad (\text{A13})$$

one can immediately obtain that

$$\begin{aligned} e^{\alpha K_z} M e^{-\alpha K_z} &= m_1 e^{\alpha K_z} K_x e^{-\alpha K_z} + m_2 e^{\alpha K_z} K_y e^{-\alpha K_z} + m_3 K_z \\ &= (m_1 \cos[\alpha] - m_2 \sin[\alpha]) K_x + (m_1 \sin[\alpha] + m_2 \cos[\alpha]) K_y + m_3 K_z \\ &= \sqrt{m_1^2 + m_2^2} K_y + m_3 K_z. \end{aligned} \quad (\text{A14})$$

Next we show that there is a matrix $P_2 = e^{\beta K_x} \in \text{Sp}_{2, \mathbb{R}}$ which can convert $\sqrt{m_1^2 + m_2^2} K_y + m_3 K_z$ into $\sqrt{2 \text{Tr}[M^2]} K_y$. Using Eq. (A4) we know that $m_1^2 + m_2^2 - m_3^2 > 0$ and so we can choose β such that

$$\sinh[\beta] = \frac{m_3}{\sqrt{m_1^2 + m_2^2 - m_3^2}}, \quad \cosh[\beta] = \frac{\sqrt{m_1^2 + m_2^2}}{\sqrt{m_1^2 + m_2^2 - m_3^2}}. \quad (\text{A15})$$

Make use of Eq. (A13) again and obtain

$$\begin{aligned} e^{\beta K_x} (\sqrt{m_1^2 + m_2^2} K_y + m_3 K_z) e^{-\beta K_x} &= \sqrt{m_1^2 + m_2^2} e^{\beta K_x} K_y e^{-\beta K_x} + m_3 e^{\beta K_x} K_z e^{-\beta K_x} \\ &= (\sqrt{m_1^2 + m_2^2} \cosh[\beta] - m_3 \sinh[\beta]) K_y + (m_3 \cosh[\beta] - \sqrt{m_1^2 + m_2^2} \sinh[\beta]) K_z \\ &= \sqrt{m_1^2 + m_2^2 - m_3^2} K_y \\ &= \sqrt{2 \text{Tr}[M^2]} K_y. \end{aligned} \quad (\text{A16})$$

Consequently the $\text{Sp}_{2,\mathbb{R}}$ matrix $e^{\beta K_x} e^{\alpha K_z}$ will convert M into $\sqrt{2} \text{Tr}[M^2] K_y$ when M is hyperbolic. \square

Using the two previous lemmas we may now proceed to a proof of Lemma 1 that was stated in the main text. First we restate it.

Lemma 1. *If Ξ only contains hyperbolic elements then Eq. (6) is similar, via a symplectic transformation, to*

$$\dot{S}(t) = (-K_x + bK_z + u(t)K_y)S(t), \quad S(0) = \mathbb{I}_2, \quad (\text{A17})$$

where b is some real constant with modulus strictly less than one.

Proof If Eq. (6) only has hyperbolic controls then the following inequality holds:

$$\text{Tr}[(A + vB)^2] = \text{Tr}[B^2]v^2 + 2\text{Tr}[AB]v + \text{Tr}[A^2] > 0, \quad (\text{A18})$$

for all $v \in \mathbb{R}$. For this inequality to hold for all v it is immediately clear that $\text{Tr}[A^2] > 0$. We can see that $\text{Tr}[B^2] > 0$ because (a) if it were less than zero then there exists v for which the inequality does not hold and (b) if it were equal to zero then $\text{Tr}[AB]$ must equal zero; by Lemma 5 this implies that $\text{Tr}([A, B]^2) = 0$ which implies that the system does not obey the Lie algebra rank criterion by Lemma 6 which would contradict our assumption.

With the knowledge that B is hyperbolic, Lemma 7 states that there exists a symplectic similarity transformation to transform Eq. (6) into:

$$\dot{S}(t) = (A' + u(t)K_y)S(t), \quad S(0) = \mathbb{I}_2, \quad (\text{A19})$$

where A' is some unspecified element of $\mathfrak{sp}_{2,\mathbb{R}}$. Expand A' in the symplectic basis of Eq. (7):

$$A' = b_x K_x + b_y K_y + b_z K_z. \quad (\text{A20})$$

By redefining $u(t)$ we can transform the system such that b_y equals zero. We know that A' is hyperbolic because this property is invariant under similarity transformation, therefore we know that $|b_x| > |b_z|$ from Eq. (A4). The role of time in Eq. (A19) allows us to rescale such that the coefficient of K_x has modulus one leaving us with system

$$\dot{S}(t) = (\epsilon K_x + bK_z + u(t)K_y)S(t), \quad S(0) = \mathbb{I}_2, \quad (\text{A21})$$

where $|b| < 1$ and $\epsilon = \pm 1$. If $\epsilon = -1$ then we leave the system as it is and the proof is finished. If $\epsilon = 1$ then enacting a similarity transformation under the symplectic matrix Ω is equivalent to time reversal and sends each of the basis matrices to their negative. Thus we have shown that Eq. (6) is symplectically similar to Eq. (A17). Note that we did not talk about effects on the initial value of X because this is set to be \mathbb{I}_2 . \square

Recalling that any 2×2 real matrix can be written as

$$X = \begin{pmatrix} x_1 + x_3 & x_2 + x_4 \\ x_4 - x_2 & x_1 - x_3 \end{pmatrix}, \quad x_i \in \mathbb{R}, \quad (\text{A22})$$

we are able to proceed to a proof of Lemma 2 stated in the main text. First we restate it.

Lemma 2. *If $X \in \tilde{\mathcal{R}}$ then the function*

$$f(X) := (x_1 - x_4)^2 - (x_2 - x_3)^2 \quad (\text{A23})$$

satisfies

$$f(X) \geq 1, \quad (\text{A24})$$

$$\frac{d}{dt}f(X) \geq 0, \quad (\text{A25})$$

$$\frac{d}{dt}f(X = \mathbb{I}) \geq 1, \quad (\text{A26})$$

for any choice of $u(t)$ in Eq. (9).

Proof Eqs. (A17) and (A22) provide the set of equations

$$\dot{x}_1 = \frac{1}{2}(ax_2 - x_4 - ux_3), \quad (\text{A27})$$

$$\dot{x}_2 = \frac{1}{2}(-ax_1 + x_3 - ux_4), \quad (\text{A28})$$

$$\dot{x}_3 = \frac{1}{2}(-ax_4 + x_2 - ux_1), \quad (\text{A29})$$

$$\dot{x}_4 = \frac{1}{2}(ax_3 - x_1 - ux_2). \quad (\text{A30})$$

Subtracting Eqs. (A27) and (A30) then followed by a succeeding multiplication by $2(x_1 - x_4)$ provides

$$\begin{aligned} \frac{d}{dt}(x_1 - x_4)^2 &= a(x_1 - x_4)(x_2 - x_3) \\ &\quad + (x_1 - x_4)^2 \\ &\quad + u(x_1 - x_4)(x_2 - x_3). \end{aligned} \quad (\text{A31})$$

Similarly, we have

$$\begin{aligned} \frac{d}{dt}(x_2 - x_3)^2 &= -a(x_1 - x_4)(x_2 - x_3) \\ &\quad - (x_2 - x_3)^2 \\ &\quad + u(x_1 - x_4)(x_2 - x_3). \end{aligned} \quad (\text{A32})$$

Then subtracting Eqs. (A31) and (A32)

$$\begin{aligned} \frac{d}{dt}((x_1 - x_4)^2 - (x_2 - x_3)^2) &= 2a(x_1 - x_4)(x_2 - x_3) \\ &\quad + ((x_1 - x_4)^2 + (x_2 - x_3)^2) \\ &= (1 - |a|)((x_1 - x_4)^2 + (x_2 - x_3)^2) \\ &\quad + |a|((x_1 - x_4) - \text{sign}(a)(x_2 - x_3))^2 \\ &\geq 0. \end{aligned} \quad (\text{A33})$$

Thus, the function f is nondecreasing for every trajectory of the system. Since the initial value of f is 1 it can be concluded that the reachable states of Eq. (A17) should satisfy the restriction that $f \geq 1$. Furthermore the initial value of the rate of change is equal to one if we set $x_1 = 1$ and $x_2 = x_3 = x_4 = 0$. This implies that the function is increasing from the beginning. \square

Appendix B: Singular value decomposition

1. Uniqueness of the singular value decomposition

To prevent any ambiguity we require that the singular value decomposition be unique. This is not true in general and therefore we need to restrict the range of allowed angles so that it is properly defined. In short, we want

$$S = R_\theta Z R_\phi = R_\alpha Z' R_\beta \quad (\text{B1})$$

to imply that $\alpha = \theta$, $\beta = \phi$ and $Z' = Z$. The first thing to notice is that the singular values of S are unique and so we would only ever get either $Z' = Z$ or $Z' = Z^{-1}$. The latter case corresponds to the situation where $z < 1$ which may be ignored provided the range of the angles is properly limited allowing $Z^{-1} = R_{-\pi/2} Z R_{\pi/2}$. Thus we need only consider two cases, $z = 1$ and $z > 1$. In the conclusion we use these cases to show that we have a freedom in how to represent the singular value decomposition.

$$a. \quad Z \neq \mathbb{I}$$

Let's first look at the former case, $Z' = Z$, where $Z \neq \mathbb{I}$. Assume a non-unique decomposition:

$$R_\theta Z R_\phi = R_\alpha Z R_\beta, \quad (\text{B2})$$

or equivalently

$$R_{\theta-\alpha} Z = Z R_{\beta-\phi}, \quad (\text{B3})$$

and explicitly

$$\begin{pmatrix} \frac{1}{z} \cos[\theta - \alpha] & -z \sin[\theta - \alpha] \\ \frac{1}{z} \sin[\theta - \alpha] & z \cos[\theta - \alpha] \end{pmatrix} = \begin{pmatrix} \frac{1}{z} \cos[\beta - \phi] & -\frac{1}{z} \sin[\beta - \phi] \\ z \sin[\beta - \phi] & z \cos[\beta - \phi] \end{pmatrix}. \quad (\text{B4})$$

This implies the set of conditions

$$\begin{aligned} \frac{1}{z} \sin[\theta - \alpha] &= z \sin[\beta - \phi], \\ z \sin[\theta - \alpha] &= \frac{1}{z} \sin[\beta - \phi], \\ \cos[\theta - \alpha] &= \cos[\beta - \phi], \end{aligned} \quad (\text{B5})$$

which only hold when

$$\begin{aligned} \sin[\theta - \alpha] &= 0, \\ \sin[\beta - \phi] &= 0, \\ \cos[\theta - \alpha] &= \cos[\beta - \phi]. \end{aligned} \quad (\text{B6})$$

These only hold when

$$\alpha = \theta + n\pi \quad \text{and} \quad \beta = \phi + m\pi \quad (\text{B7})$$

for $n, m \in \mathbb{Z}$ either both odd or both even.

To avoid Eq. (B7) being satisfied for $m, n \neq 0$ we limit ϕ to vary in a range less than π so that $\beta = \phi$. This sets $m = 0$ and so to satisfy Eq. (B7) without letting $\alpha = \theta$ the nearest option would be to let $\alpha = \theta \pm 2\pi$. The maximum range for the angles governing $\text{SO}(2)$ is 2π and so this is the bound that will apply to θ . For uniqueness, therefore, we set the ranges of θ and ϕ to:

$$-\pi + \theta_0 \leq \theta < \pi + \theta_0, \quad -\frac{\pi}{2} + \phi_0 \leq \phi < \frac{\pi}{2} + \phi_0, \quad (\text{B8})$$

where θ_0, ϕ_0 fix the centre of the ranges.

$$b. \quad Z = \mathbb{I}$$

In this case we consider $Z = \mathbb{I}$. We look for times when

$$R_\theta R_\phi = R_\alpha R_\beta \quad (\text{B9})$$

is satisfied.

These are cases when

$$\theta + \phi = \alpha + \beta + 2n\pi, \quad (\text{B10})$$

for $n \in \mathbb{Z}$.

This holds true for a whole range of angles. We can arbitrarily set $\phi = \phi_0$ to let θ label the elements of $\text{SO}(2)$.

$$c. \quad \text{Angle limit}$$

Now we have choices on how to set the angles such that the decomposition is unique. We choose

$$-\pi + \theta_0 \leq \theta < \pi + \theta_0, \quad -\frac{\pi}{2} + \phi_0 \leq \phi < \frac{\pi}{2} + \phi_0 \quad (\text{B11})$$

to make the singular value decomposition unique when $Z \neq \mathbb{I}$. θ_0 and ϕ_0 are some constants that we are free to set. Note that we have made a further arbitrary choice in exactly where to make the bounds tight. For $Z = \mathbb{I}$ we must totally restrict one of the angles and leave the other free; we choose so set $\phi = \phi_0$.

2. Singular value decomposition coordinates for f

In this section we represent $\cos[\theta]$ as $c\theta$ and $\sin[\theta]$ as $s\theta$ for brevity. We begin with two expressions for $X \in \text{Sp}_{2,\mathbb{R}}$:

$$X = \begin{pmatrix} x_1 + x_3 & x_2 + x_4 \\ x_4 - x_2 & x_1 - x_3 \end{pmatrix}, \quad (\text{B12})$$

and

$$X = \begin{pmatrix} \frac{c\theta c\phi}{s\theta c\phi} - z s\theta s\phi & -\frac{c\theta s\phi}{s\theta c\phi} - z s\theta c\phi \\ \frac{s\theta c\phi}{z} + z c\theta s\phi & -\frac{s\theta s\phi}{z} + z c\theta c\phi \end{pmatrix}. \quad (\text{B13})$$

Equating the two expression and solving for x_i we find that and so

$$2x_1 = \frac{1}{z}(c\theta c\phi - s\theta s\phi) + z(c\theta c\phi - s\theta s\phi), \quad (\text{B14})$$

$$2x_2 = -\frac{1}{z}(s\theta c\phi + c\theta s\phi) - z(s\theta c\phi + c\theta s\phi), \quad (\text{B15})$$

$$2x_3 = \frac{1}{z}(c\theta c\phi + s\theta s\phi) - z(s\theta s\phi + c\theta c\phi), \quad (\text{B16})$$

$$2x_4 = \frac{1}{z}(s\theta c\phi - c\theta s\phi) + z(c\theta s\phi - s\theta c\phi), \quad (\text{B17})$$

$$2(x_1 - x_4) = \frac{1}{z}(c\theta c\phi - s\theta s\phi - s\theta c\phi + c\theta s\phi) + z(c\theta c\phi - s\theta s\phi - c\theta s\phi + s\theta c\phi), \quad (\text{B18})$$

$$2(x_2 - x_3) = -\frac{1}{z}(s\theta c\phi + c\theta s\phi + c\theta c\phi + s\theta s\phi) - z(s\theta c\phi + c\theta s\phi - s\theta s\phi - c\theta c\phi), \quad (\text{B19})$$

or more simply

$$\begin{aligned} 2(x_1 - x_4) &= \frac{1}{z}(c\theta - s\theta)(c\phi + s\phi) + z(c\theta + s\theta)(c\phi - s\phi), \\ 2(x_2 - x_3) &= -\frac{1}{z}(c\theta + s\theta)(c\phi + s\phi) - z(c\theta - s\theta)(-c\phi + s\phi), \end{aligned} \quad (\text{B20})$$

which leads to

$$\begin{aligned} (x_1 - x_4)^2 &= \frac{1}{4} \left(\frac{1}{z^2}(c\theta - s\theta)^2(c\phi + s\phi)^2 + z^2(c\theta + s\theta)^2(c\phi - s\phi)^2 + 2(c\theta + s\theta)(c\theta - s\theta)(c\phi + s\phi)(c\phi - s\phi) \right), \\ (x_2 - x_3)^2 &= \frac{1}{4} \left(\frac{1}{z^2}(c\theta + s\theta)^2(c\phi + s\phi)^2 + z^2(c\theta - s\theta)^2(c\phi - s\phi)^2 - 2(c\theta + s\theta)(c\theta - s\theta)(c\phi + s\phi)(c\phi - s\phi) \right). \end{aligned} \quad (\text{B21})$$

Subtracting the two

$$\begin{aligned} (x_1 - x_4)^2 - (x_2 - x_3)^2 &= \frac{1}{4} \left(\frac{1}{z^2}(c\phi + s\phi)^2((c\theta - s\theta)^2 - (c\theta + s\theta)^2) + \right. \\ &\quad \left. z^2(c\phi - s\phi)^2((c\theta + s\theta)^2 - (c\theta - s\theta)^2) + \right. \\ &\quad \left. 4(c\theta^2 - s\theta^2)(c\phi^2 - s\phi^2) \right), \end{aligned} \quad (\text{B22})$$

to

$$(x_1 - x_4)^2 - (x_2 - x_3)^2 = \frac{1}{4} \left(\frac{1}{z^2}(1 + s2\phi)(-s2\theta) + z^2(1 - s2\phi)(s2\theta) + 4c2\theta c2\phi \right), \quad (\text{B23})$$

to

$$(x_1 - x_4)^2 - (x_2 - x_3)^2 = c2\theta c2\phi - s2\theta \left(\frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) s2\phi - \frac{1}{2} \left(z^2 - \frac{1}{z^2} \right) \right). \quad (\text{B24})$$

which is our new expression for f in terms of θ , ϕ and z .

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